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THE THEORY OF FUNCTIONS OF A REAL VARIABLE.

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By Dr. E. W. Hobson, F.R.S. Pp. xvi+772. (Cambridge: University Press, 1907.) Price 21s. net.

IT is impossible to read Dr. Hobson's book without reflecting on the marvellous change that has come over Cambridge mathematics in the last twenty years. Twenty years ago Cambridge mathematics was a thing standing by itself, and with its own virtues and defects. Pure mathematics in Cambridge meant Cayley and a few disciples; and Cayley (widely as he read) owed little or nothing to anyone but himself. Certainly he never appreciated the most fundamental ideas of modern Continental analysis. It is probable that he could not have defined a function or a limit in a way which would have satisfied Weierstrass or Dr. Hobson: it is certain that he would have been as incapable as any Senior Wrangler of proving any of the less obvious theorems of convergence. The first signs of the absorption of these ideas are to be found, not in Cayley, but in Stokes.

Now Cambridge has fallen into line. There are no Cayleys, perhaps, but there is quite a flourishing school of pure mathematics, working by what may be called German methods and on German lines, and making up in numbers and soundness for anything that it has lost in distinction. The school of Cayley is dead, and so (what is perhaps even more to be regretted) is the old Cambridge school of applied mathematics: pure mathematics and experiment have combined to kill it, and the Stokes Lecturer in Applied Mathematics writes books like this. We wonder what Clerk Maxwell or even Stokes himself would have thought of it.

However, all this is not Dr. Hobson's fault, and we must not blame him if the reflections which it inspires are not altogether pleasant. And we hasten to congratulate him on the completion of what is, without a doubt, a magnificent piece of work. It would be a fine piece of work even if were a mere compilation; for the subject is one of which there was no systematic account in English, and which no previous English writer had ever really mastered. But the book is far from being a compilation, for Dr. Hobson has made the subject his own, and writes with the air of mastery that only original work can give: and even in French, German, or Italian, there is no book which covers anything like the same ground. Dini (whom Dr. Hobson has obviously taken as his model) has held the field for a long time, and Dr. Hobson can fairly claim to have superseded him.

In taking Dini as his model, Dr. Hobson has made the "theoretically general," rather than what Borel has called the "practically general," his goal. No doubt he had to make his choice, but we must confess that he seems to us to have gone too far. Let us consider his treatment of "double limit problems," for example, problems such as those of the differentiation or integration of an infinite series or an infinite integral (why will he persist in making the uninitiated scoff by his fondness for the word "improper"?). Such problems may be approached from two different points of view. We may ask, "What is absolutely the most general form in which we can state our theorems, when we utilise all the most modern theories of sets of points, Lebesgue integrals, and the like?" This is the point of view of Dini and Dr. Hobson. On the other hand, we may ask, "In what special forms do these problems naturally occur in analysis? What are *the really important cases*? Can we state our theorems in such a way that writers on applied mathematics, or other branches of pure mathematics, when they are confronted, as they continually are, with particular problems of this kind in all conscience difficult enough, will be able to turn to us for a solution of their difficulties?" These questions must be continually before us, if we are aiming at Borel's "practically general" completeness, and even an author who has decided to aim at the other ideal will do well to keep them clearly in sight; and we wish that Dr. Hobson had more often adopted this point of view. He might then have made his book a good deal more useful and attractive for the ordinary worker in the fields of analysis. The latter, as it is, is likely to find himself faced by many theoretical difficulties to which he will not easily find an answer in Dr. Hobson's pages. However, it is perhaps as well that Dr. Hobson should leave something for someone else to do.

But it is time that we said a little of the details of the book. It is needless to say that it is beautifully and almost faultlessly printed. It is a pity, though, that the chapters are so long. Long chapters do not make a difficult book easier to read, nor do they make it easier for the author to arrive at the proper logical arrangement of the subject-matter—as appears very clearly in chapters v. and vi., which had much better have been broken up into half a dozen shorter chapters. We should like to have seen a great many more examples. Summaries of the chapters, too, would have been useful; and the author is too sober in his use of different kinds of type. In a word, he shows too great a contempt for the arts of popularity.

There are seven chapters in all. For the first three, which are of a particularly abstract character, we have practically nothing but praise. The matter is admirably selected and admirably arranged, and Dr. Hobson writes with a lucidity and distinction rare indeed among mathematicians. Nothing could be better in its way, for example, than his terse criticism of the "formal" view of mathematics (pp. 9–10). We cannot entirely agree with the conclusions at which he arrives in the course of the critical discussions of chapter iii, but we can appreciate the clear and temperate manner of his criticisms, advanced, as he says, "with some diffidence, on account of the great logical difficulties of the subject," and in the hope that "they may be of utility as a contribution towards the discussion of questions of great interest which, at the present time, cannot be regarded as having been decisively settled."

In chapter iv. we begin for the first time to be

bored in places. The four derivatives of a function *are* dull, and no one will ever make them seem anything else; and a good deal of Brodén's work is much more solid than inspiring. Occasionally we do not quite like Dr. Hobson's choice of words—in particular we may instance his use of "indefinitely great," in such phrases as "has indefinitely great values," "the functional value is regarded as indefinitely great," "the lower limit is indefinitely great." Why not, in the last case, simply "there is no lower limit"? Dr. Hobson could reply that he has expressly warned the reader against any such confusion of thought as is sometimes implied in modes of expression such as these; and there is certainly none in his own mind. None the less we wish that he had expressed himself in a different manner.

In this chapter, let us single out for special praise the sections on double and repeated limits (pp. 303 *et seq.*). We particularly like the author's generalisation of the definition of a repeated limit, which enables him to simplify the statements of a number of theorems. We have already said that we do not altogether like the arrangement of the next two chapters. Surely it would have been better to introduce the notion of a series at an earlier stage. As it is, some of the theorems concerning integrals are separated from one another in a rather irritating way. But most of the discussions of particular theorems are admirable. We may mention especially the treatment of the "absolutely convergent improper integral" (pp. 364 *et seq.*), the sections on the transformation of double integrals (pp. 445 *et seq.*), and the account of Baire's theory of the representation of functions (pp. 522 *et seq.*). A few criticisms of details suggest themselves. Is it worth while to define "principal values" if nothing more is to be said about them? There is a curious slip on p. 454, l. 14; obviously the condition stated is not *necessary:* and it is very odd that Dr. Hobson should define *divergence* and *oscillation* in such a way that $1-2+3-4+\ldots$ is a divergent rather than an oscillating series. The last word has not yet been said about Weierstrass's non-differentiable function (pp. 620 *et seq.*). What about $\Sigma a^n \cos b^n x$, where ab is only a little greater than 1? One would expect the function to have no differential coefficient whenever $ab \geqq 1$; but no one seems to have found out whether this is the case or not.

Finally, chapter vii. (Fourier's Series) shows Dr. Hobson quite at his best. The last part, in which he supplies a final touch of rigour to some of Riemann's work, is extremely difficult, but that was inevitable. The remark at the foot of p. 647 is open to dispute. Was not something very much like the theorem, ascribed to Lerch on p. 727, also proved by Stokes? On p. 732, l. 24, for "diminished" read "increased."

A short appendix contains some further critical remarks, in addition to chapter iii. We wish that there had been space for a summary of König's rather watery theories, and the author's neat and convincing reply in the London Mathematical Society's Proceedings. We must confess to a strong temptation to argue with Dr. Hobson concerning the remarks at the top of p. 765, but the temptation must be resisted.

Dr. Hobson has attempted an appalling task. There is no region of pure mathematics (unless it be the theory of numbers) which is quite so difficult as this; certainly none of which the literature is so scattered and so difficult to collate, or in which the writing of a big book requires a greater combination of drudgery and critical insight. All things considered, he has succeeded wonderfully. We can think of no one else who would have done half as well. G. H. H.

LIEBIG AND GÜSSEFELD.

Justus von Liebig und Emil Louis Ferdinand Güssefeld. Briefwechsel: 1862–1866. Herausgegeben von Dr. O. E. Güssefeld. Pp. viii+72. (Leipzig: Johann Ambrosius Barth, 1907.) Price 3 marks.

THIS little book has a twofold interest. To the scientific agriculturist it is interesting as elucidating the history of the introduction of the modern methods of agriculture into Germany, and especially of the introduction of the so-called chemical fertilisers, due largely to the teaching and influence of Liebig; it serves also to throw some sidelights upon the character and habits of Liebig himself, and is therefore of interest to the historian of chemistry. It consists simply of a collection of thirty-eight letters which passed between Liebig and Emil Güssefeld from 1862 to 1866, twenty-two of which are contributed by Liebig, and the whole has been arranged for publication, with explanatory notes and annotations, by the pious care of the son of one of the correspondents.

Emil Güssefeld was a Hamburg merchant, of the conventional type, dealing mainly in coffee and other colonial products. In a fortunate hour he accepted an agency from an American company for the sale in Germany of guano from Baker Island, in the Pacific Ocean, and thereby laid the foundations of a prosperous business in phosphatic manures. Emil Güssefeld indeed stands to Germany in much the same relation that the late Sir John Bennett Lawes stands to this country, and both reaped fame and fortune by the far-sighted enterprise which induced them to give practical effect to the theoretical views of Liebig. As a prudent man, Güssefeld, before undertaking the agency, seems to have consulted Liebig as to the probability that the Baker guano, of the merits of which he was well assured, would find a ready sale among a body of agriculturists who are even more conservative than our own, and Liebig's reply constitutes the first letter in the series. It is in every respect worthy of him—sound, thoughtful, and considerate, and with that note of cautious optimism which the eminently practical mind of the Hamburg merchant could not fail to appreciate. Liebig, as this correspondence abundantly testifies, never spared himself when his interest was aroused, and he was ever ready to give of his best, without fee or thought of reward, when the object commended itself to him. In this large-hearted liberality Liebig resembled Davy, who nearly half a century previously had striven in the same self-sacrificing way to infuse something of the scientific spirit into the oldest of the arts. Liebig's letters are rich in practical advice, business hints, analytical information